A Revised Routh Approximations for Interval System

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Abstract: By using Routh approximation method and related reduction method to deal with the model reduction of high-order systems, it is important to preserve the stability. This note proposes an effective algorithm to modify those methods given before. Our method is not conservative when we reduce a stable high-order interval system to a stable lower-order interval system. This is illustrated by several numerical examples.

Key words: Interval System; Interval Reduction; Interval Arithmetic; Monotonicity; Routh Approximation;

1 Introduction

Among many system reduction methods, Routh-Pade approximation method is remarkable for lots of useful features such as the computational simplicity, matching of 2r time moments and preserving the stability [1, 2, 3]. This method is introduced into interval systems [4, 5, 6]. Hwang and Yang[7] show that the advantage of preserving stability of Routh approximation is invalid when it is used by direct truncation of Routh table of a stable higher order polynomial. Therefore, some works have been done to solve this problem [8, 9, 10]. It is shown that it is possible to obtain an unstable interval Routh approximation for a stable original interval system. Then a revised method of direct Routh table truncation for model reduction of interval system is proposed in [8], and it claims that the resulting reduced order interval polynomial is guaranteed to be stable. But an example given by Yang[9] shows that it is not correct. And it is pointed in [9] that the principle reason of no preserving stability is that the interval arithmetic operations are irreversible. As a reply, Dolgin[10] formulates two additional conditions to avoid the situation of stability loss. This makes their method being not operative, concise and computational simplicity. Reference [11] proposes an improvement to $\gamma - \delta$ Routh approximation technique for interval systems using the Kharitonov polynomials. Its advantage is that it can guarantee the stability of reduced models if the original system is stable. Its disadvantage

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is that the proposed improved technique is based on Kharitonov robust stability theory, and the Routh table is constructed by extracting from the over bounded Routh table using Kharitonov polynomials with the optimization techniques. That is to say, the result is an optimization method which revised the Routh array through Kharitonov polynomials. This method is not concise and clear, but Routh approximation is.

It is observed if the system has eigenvalue which is close to imaginary axis, whether a stable reduced order interval model can be derived from an interval system by using Routh-Pade approximation method has great relationship to the algorithm. In this case the element in Routh array of the interval polynomial is very sensitive to the coefficients of polynomial and is closely dependent on algorithm. Based on those points, we propose a new revised algorithm, which is concise, easy to compute and keep better preserving stability than those given before.

2 Main Result

Consider transfer function,

$$G(s) = \frac{[a_0^-, a_0^+] + [a_1^-, a_1^+]s + \cdots + [a_{n-1}^-, a_{n-1}^+]s^{n-1}}{[b_0^-, b_0^+] + [b_1^-, b_1^+]s + \cdots + [b_n^-, b_n^+]s^n} = \frac{N(s)}{D(s)} \tag{1}$$

Let $[a, b]$ and $[c, d]$ be two intervals, and define

Addition: $[a, b] + [c, d] = [a + b, c + d]$

Subtraction: $[a, b] - [c, d] = [a - d, b - c]$

Multiplication: $[a, b][c, d] = [min\{ac, ad, bc, bd\}, max\{ac, ad, bc, bd\}]$

Division: $\frac{[a,b]}{[c,d]} = [a, b][\frac{1}{d}, \frac{1}{c}]$, provided $0 \notin [c, d]$

What following is the introduction of our revised Routh approximation method for interval systems. Let $G_r(s)$ be the reduced interval systems of degree r, i.e.

$$G_r(s) = \frac{[f_0^-, f_0^+] + [f_1^-, f_1^+]s + [f_2^-, f_2^+]s^2 + \cdots + [f_{r-1}^-, f_{r-1}^+]s^{r-1}}{[d_0^-, d_0^+] + [d_1^-, d_1^+]s + [d_2^-, d_2^+]s^2 + \cdots + [d_r^-, d_r^+]s^r} \triangleq \frac{N_r(s)}{D_r(s)} \tag{2}$$

The Routh table for the denominator $D(s)$ of the interval systems is given below,

$$\begin{array}{ccccc}
[b_{1,1}^-, b_{1,1}^+] & [b_{1,2}^-, b_{1,2}^+] & [b_{1,3}^-, b_{1,3}^+] & [b_{1,4}^-, b_{1,4}^+] & \cdots \\
[b_{2,1}^-, b_{2,1}^+] & [b_{2,2}^-, b_{2,2}^+] & [b_{2,3}^-, b_{2,3}^+] & \cdots & \\
[b_{3,1}^-, b_{3,1}^+] & [b_{3,2}^-, b_{3,2}^+] & [b_{3,3}^-, b_{3,3}^+] & \cdots & \\
[b_{4,1}^-, b_{4,1}^+] & [b_{4,2}^-, b_{4,2}^+] & \cdots & & \\
\vdots & \cdots & & & \\
[b_{i,1}^-, b_{i,1}^+] & [b_{i,2}^-, b_{i,2}^+] & \cdots & & \\
[b_{i+1,1}^-, b_{i+1,1}^+] & [b_{i+1,2}^-, b_{i+1,2}^+] & \cdots & & \\
\vdots & \cdots & & &
\end{array}$$

Table 1

where

$$[b_{1,1}^-, b_{1,1}^+] = [b_n^-, b_n^+],\ \ [b_{1,2}^-, b_{1,2}^+] = [b_{n-2}^-, b_{n-2}^+],\ \ [b_{1,3}^-, b_{1,3}^+] = [b_{n-4}^-, b_{n-4}^+] \cdots\cdots,$$
$$[b_{2,1}^-, b_{2,1}^+] = [b_{n-1}^-, b_{n-1}^+],\ \ [b_{2,2}^-, b_{2,2}^+] = [b_{n-3}^-, b_{n-3}^+],\ \ [b_{2,3}^-, b_{2,3}^+] = [b_{n-5}^-, b_{n-5}^+] \cdots\cdots.$$

And the above array is formed by algorithm (3), i.e.

$$b_{i,j} = \frac{b_{i,j+1}b_{i-1,1} - b_{i-2,1}b_{i-1,j+1}}{b_{i-1,1}} \tag{3}$$

where $b_{i,j} \in [b_{i,j}^-, b_{i,j}^+]$, $i \geq 3$, $1 \leq j \leq \frac{n-i+3}{2}$.

Then the denominator of $G_r(s)$ is given by

$$D_r(s) = b_{r,1}s^r + b_{r-1,1}s^{r-1} + b_{r,2}s^{r-2} + b_{r-1,2}s^{r-3} + \cdots\cdots \tag{4}$$

Remark 1 : The numerator of $G_r(s)$ is given by matching the coefficients of the power series expansion of the system and the reduced model[4].

If we directly used the interval arithmetic to derive Routh stability array, then intervals of array(Table 1) given by (3) is enlarged. This point can be seen from the following points:

1) All entries in each row($i \geq 3$) are computed by using an interval, not a point;

2) Interval arithmetic can not ensure the consistency of each pair of rows and columns;

3) Interval arithmetic is sensitive to the computational algorithm, such as

$$b_{i-2,j+1} - \frac{b_{i-2,1}b_{i-1,j+1}}{b_{i-1,1}} \subseteq \frac{b_{i-2,j+1}b_{i-1,1} - b_{i-2,1}b_{i-1,j+1}}{b_{i-1,1}}$$

while they are same for four given numbers.

In this paper, we focus on keeping the consistency of elements of each pair of rows and columns with interval arithmetic.

Firstly, it should be pointed out that $b_{i,j}$ is increasing on $b_{i-1,1}$ and decreasing on $b_{i-2,1}b_{i-1,j+1}$, where $b_{i,j}$ is given by

$$b_{i,j} = b_{i-2,j+1} - \frac{b_{i-2,1}b_{i-1,j+1}}{b_{i-1,1}} \tag{5}$$

Let $\triangle_{i,j} \triangleq [\triangle_{i,j}^-, \triangle_{i,j}^+]$ and $\nabla_{i,j} \triangleq [\nabla_{i,j}^-, \nabla_{i,j}^+]$ be the left and right intervals corresponding to $b_{i,j}$. For $i \geq 3$, $1 \leq j \leq \frac{n-i+3}{2}$, our revised Routh array is computed by

$$\triangle_{i,j} = \begin{cases} \triangle_{i-2,j+1} - \dfrac{\triangle_{i-2,1}^+\triangle_{i-1,j+1}^+}{\triangle_{i-1,1}^-} & i \geq 3, i \neq 4 \\ \triangle_{i-2,j+1}^+ - \dfrac{\triangle_{i-2,1}^-\triangle_{i-1,j+1}^+}{\triangle_{i-1,1}^-} & i = 4 \end{cases} \tag{6}$$

$$\nabla_{i,j} = \begin{cases} \nabla_{i-2,j+1} - \dfrac{\nabla_{i-2,1}^-\nabla_{i-1,j+1}^-}{\nabla_{i-1,1}^+} & i \geq 3, i \neq 4 \\ \nabla_{i-2,j+1}^- - \dfrac{\nabla_{i-2,1}^+\nabla_{i-1,j+1}^-}{\nabla_{i-1,1}^+} & i = 4 \end{cases} \tag{7}$$

and

$$\triangle_{i,j} = \nabla_{i,j} = [b_{i,j}^-, b_{i,j}^+] \qquad (i = 1, 2; 1 \leq j \leq n) \tag{8}$$

$$\begin{cases} \triangle_{4,j} = [\triangle_{4,j}^-, \triangle_{4,j}^+] & if \quad \triangle_{4,j}^- = \triangle_{4,j}^+ \\ \nabla_{4,j} = [\nabla_{4,j}^-, \nabla_{4,j}^+] & if \quad \nabla_{4,j}^- = \nabla_{4,j}^+ \end{cases} \tag{9}$$

Therefore, our revised Routh array algorithm is computed row by row in the following fashion, which will be referred to as algorithm RRA.

Step 1, evaluate the first two rows of Routh array by (8);

Step 2, compute other rows by (6) and (7);

Step 3, put $b_{i,j} = [\triangle_{i,j}^-, \nabla_{i,j}^+]$, $i \geq 3$, $1 \leq j \leq \frac{n-i+3}{2}$;

Step 4, if $\triangle_{i,j}^- \leq \nabla_{i,j}^+$ for every interval element $[\triangle_{i,j}^-, \nabla_{i,j}^+]$, then end;

Step 5, otherwise, then turn to step 4.

3 Examples

Example 1: Consider the family of seven order interval polynomial[10]

$$D(s) = [1,2]s^7 + [9,10]s^6 + [31,35]s^5 + [71,72]s^4 + [111,112]s^3 + [109,110]s^2 + [76,84]s + [12,13] \tag{10}$$

Firstly, the Routh table given by Dolgin's method in [8] is

$[1,2]$	$[31,35]$	$[111,112]$	$[76,84]$
$[9,10]$	$[71,72]$	$[109,110]$	$[12,13]$
$[19.63,23.79]$	$[93.63,94.79]$	$[73.95,82.11]$	
$[29.52,31.03]$	$[73.07,77.64]$	$[12.00,13.00]$	
$[37.95,42.39]$	$[64.63,73.50]$		
$[17.68,28.94]$	$[12.00,13.00]$		
$[42.22,52.82]$			

Table 2

Secondly, the Routh table given by Dolgin's method in [10] is

$[1,2]$	$[31,35]$	$[111,112]$	$[76,84]$
$[9,10]$	$[71,72]$	$[109,110]$	$[12,13]$
$[19.79,23.63]$	$[93.86,94.56]$	$[77.73,78.33]$	
$[29.93,30.62]$	$[75.16,75.56]$	$[12.33,12.68]$	
$[39.97,40.38]$	$[68.95,69.18]$		
$[23.19,23.42]$	$[12.46,12.54]$		
$[47.48,47.56]$			

Table 3

Thirdly, the Routh table given by algorithm RRA is

$[1,2]$	$[31,35]$	$[111,112]$	$[76,84]$
$[9,10]$	$[71,72]$	$[109,110]$	$[12,13]$
$[15.00,27.90]$	$[86.56,101.10]$	$[73.11,82.80]$	
$[19.47,335.12]$	$[61.33,82.19]$	$[12,13]$	
$[26.69,45.17]$	$[60.42,74.63]$		
$[11.43,30.38]$	$[12,13]$		
$[28.94,57.19]$			

Table 4

It can be observed from Tables 2,3,4 that,

1) all of reduced-order interval models are stable;

2) the interval elements in Table 4 are larger than those in Table 2 and Table 3;

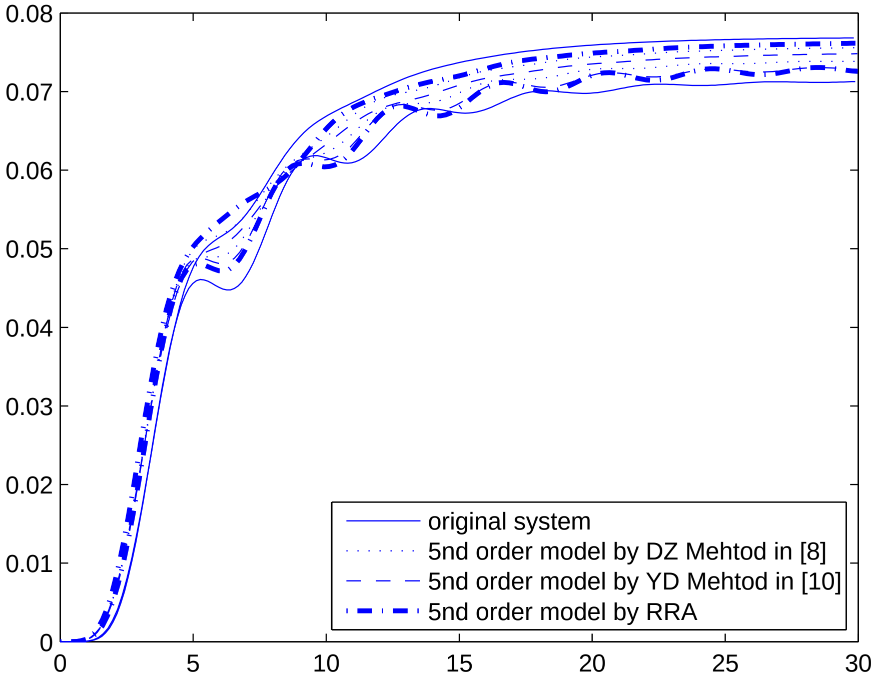
The fifth-order reduced interval polynomials yielded by [8], [10] and algorithm RRA, respectively, are

$$P_5(s) = [19.63,23.79]s^5 + [29.52,31.03]s^4 + [93.63,94.79]s^3 + [73.07,77.64]s^2 + [73.95,82.11]s + [12,13]$$

$$R_5(s) = [19.79,23.63]s^5 + [29.93,30.62]s^4 + [93.86,94.56]s^3 + [75.10,75.56]s^2 + [77.73,78.33]s + [12.33,12.68]$$

$$D_5(s) = [15.00,27.90]s^5 + [19.47,35.12]s^4 + [86.56,101.10]s^3 + [61.33,82.19]s^2 + [73.11,82.80]s + [12,13]$$

The robust step responses of uncertain systems $\frac{1}{D(s)}$, $\frac{1}{P_5(s)}$, $\frac{1}{R_5(s)}$ and $\frac{1}{D_5(s)}$ are shown in the following figure:



The following two examples shown in [7] and [10] are reworked here.

Example 2: Consider a six order interval polynomial

$$D(s) = [2, 2.5]s^6 + [76, 76.5]s^5 + [119, 119.5]s^4 + [111, 111.5]s^3 + [71, 71.5]s^2 + [31, 31.5]s + [9, 9.5]$$

Routh tables derived by method of reference [10] and by algorithm RRA, respectively, are

Routh Table of using method in [10]				**Routh Table of using algorithm RRA**			
$[2, 2.5]$	$[119, 119.5]$	$[71, 71.5]$	$[9, 9.5]$	$[2, 2.5]$	$[119, 119.5]$	$[71, 71.5]$	$[9, 9.5]$
$[76, 76.5]$	$[111, 111.5]$	$[31, 31.5]$		$[76, 76.5]$	$[111, 111.5]$	$[31, 31.5]$	
$[115.72, 116.21]$	$[70.18, 70.47]$	$[9.1, 9.4]$		$[115.33, 116.60]$	$[69.96, 70.69]$	$[9.00, 9.50]$	
$[64.86, 65.16]$	$[25.12, 25.22]$			$[64.95, 65.07]$	$[25.10, 25.24]$		
$[25.38, 25.48]$	$[9.24, 9.27]$			$[25.03, 25.83]$	$[9.00, 9.50]$		
$[1.51, 1.54]$				$[0.55, 2.47]$			
$[9.24, 9.27]$				$[9.00, 9.50]$			

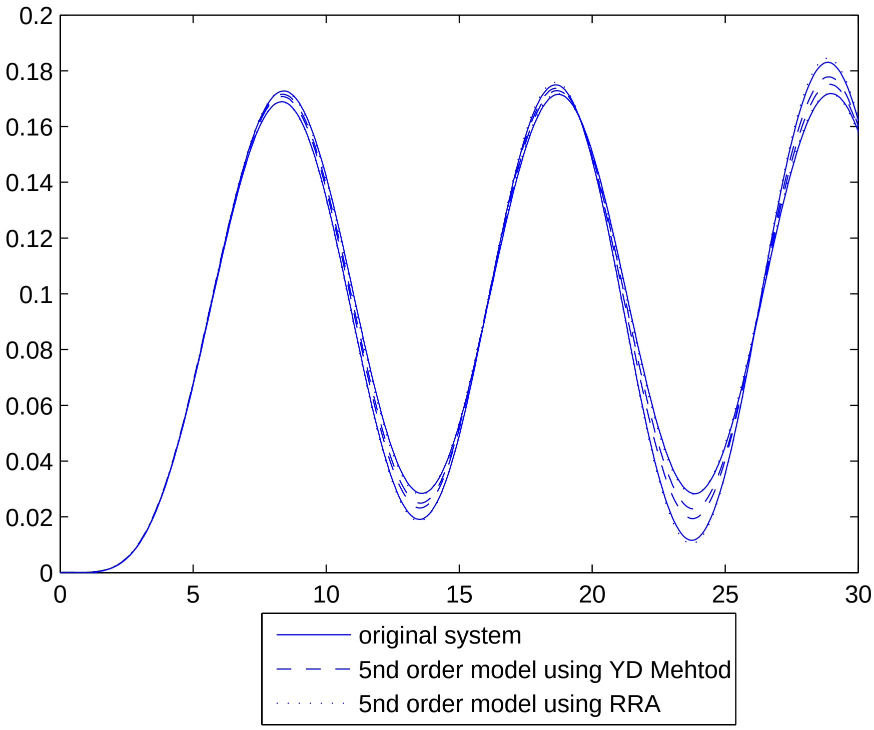
Table 5

Truncating one row in Table 5 results in the following two sixth-order polynomials:

$$P_5(s) = [76, 76.5]s^5 + [115.72, 116.21]s^4 + [111, 111.5]s^3 + [70.18, 70.47]s^2 + [31, 31.5]s + [9.1, 9.4]$$

$$D_5(s) = [76, 76.5]s^5 + [115.33, 116.60]s^4 + [111, 111.5]s^3 + [69.96, 70.69]s^2 + [31, 31.5]s + [9, 9.5]$$

The robust step responses of uncertain systems $\frac{1}{D(s)}$, $\frac{1}{P_5(s)}$ and $\frac{1}{D_5(s)}$ are shown below,



Example 3: Consider a six order interval polynomial

$$D(s) = [2.1, 2.6]s^6 + [76.1, 76.7]s^5 + [119.1, 119.6]s^4 + [111, 111.6]s^3 + [71.8, 72.3]s^2 + [31, 31.7]s + [9, 9.9]$$

Routh tables of $D(s)$ using DZ method in [8] and algorithm RRA, respectively, are

Routh Table of using DZ method in [8]				**Routh Table of using algorithm RRA**			
$[2.1, 2.6]$	$[119.1, 119.6]$	$[71.8, 72.3]$	$[9, 9.9]$	$[2.1, 2.6]$	$[119.1, 119.6]$	$[71.8, 72.3]$	$[9, 9.9]$
$[76.1, 76.6]$	$[111, 111.6]$	$[31, 31.7]$		$[76.1, 76.6]$	$[111, 111.6]$	$[31, 31.7]$	
$[115.67, 116.19]$	$[70.82, 71.35]$	$[9.00, 9.90]$		$[115.29, 116.56]$	$[70.17, 71.45]$	$[9.00, 9.90]$	
$[63.98, 64.92]$	$[24.48, 25.77]$			$[64.31, 64.59]$	$[25.07, 25.17]$		
$[24.48, 27.32]$	$[9.00, 9.90]$			$[25.60, 26.19]$	$[9.00, 9.90]$		
$[-0.16, 3.37]$				$[0.19, 2.98]$			
$[9.00, 9.90]$				$[9.00, 9.90]$			

Table 6

It can be observed easily from Table 6 that the reduced-order interval polynomial of $D(s)$ derived by DZ method is not stable, and derived by our algorithm RRA is stable while the original interval polynomial is stable.

Remark 2 : It is observed from examples 1 and 2 that almost every interval element $[b_{i,j}^-, b_{i,j}^+]$ in the Routh table of using DZ method in [8] is smaller than that in the Routh table of using algorithm RRA, and observed from example 3 that the interval entry $b_{6,1} = [-0.16, 3.17]$ in the Routh table of using DZ method in [8] contains zero. The two computed interval Routh arrays shown in table 6 indicate that the method given in [8] cannot guarantee the success in generating a full interval Routh array, and our algorithm RRA can do.

Remark 3 : If one polynomial of interval polynomials has complex roots near the imaginary axis, then the element in Routh array of the interval polynomial is very sensitive to the coefficients of polynomial and is closely dependent on algorithm.

4 Conclusions

The advantage of Routh approximation is to yield stable reduced-order model for stable original high-order systems. However, the advantage of preservation of the Routh approximation method is invalid when it is extended to the model reduction of interval systems. To overcome this, a revised Routh algorithm is proposed in this note for the model reduction of high-order interval systems. With our revised algorithm, a stable reduced-order interval model can be obtained as possible as. At the same time, almost every element in the Routh table is as large as possible.

References

[1] M.F.Hutton and B.Friedland, Routh Approximations for Reducing Order of Linear,Time-Invariant Systems, *IEEE Trans. Automat. Contr.Vol.AC.*, **20**:3(1975), pp.329–337.

[2] Y.shamash, Model reduction using the Routh stability criterion and the Pade approximation technique, *Int. J. Contr.Vol.*, **21**:3(1975), pp.475–484.

[3] A.Lepachy and U.viara,, An improvement in the Routh-Pade approximation techniques, *Int. J. Contr.Vol.*, **36**:4(1982), pp.643–661.

[4] B.Bandyopadhyay, O.Ismail and R.Gorez, Routh-Pade Approximation for Interval Systems, *IEEE Trans. Automat. Contr.Vol.*, **39**:12(1994), pp.2454–2456.

[5] B.Bandyopadhyay, Avinash Upadhye and Osman Ismail, $\gamma - \delta$ Routh Approximation for Interval Systems, *IEEE Trans. Automat. Contr.Vol.*, **42**:8(1997), pp.1127–1130.

[6] O.Ismil, B.Bandyopadhyay and R.Gorez, Discrete Interval Systems reduction using Pade Approximation to Allow Retention of Dominant Poles, *IEEE Trans. Circuits Syst.*, **44**:11(1997), pp.1075–1078.

[7] Chyi Hwang and Shih-Feng Yang, Comments on the Computation of Interval Routh Approximations, *IEEE Trans. Automat. Contr.Vol.*, **44**:9(1999), pp.1782-1787.

[8] Yuri Dolgin and Ezra Zeheb, On Routh-Pade Model Reduction of Interval Systems, *IEEE Trans. Automat. Contr.Vol.*, **48**:9(2003), pp.1610–1612.

[9] Shih-Feng Yang, Comments on "On Routh-Pade Model Reduction of Interval Systems", *IEEE Trans. Automat. Contr.Vol.*, **50**:2(2005), pp.273–274.

[10] Yuri Dolgin, Author's Reply, *IEEE Trans. Automat. Contr.Vol.*, **50**:2(2005), pp.274-275.

[11] B. Bandyapadhyay, V. Sreeram and P. Shingare, Stability $\gamma - \delta$ Routh Approximation for Interval Systems Using Kharitonov Polynomials, *Int. J. Inf. Syst. Sci., Vol.*, **4**:3(2008), pp.348-361.